REFERENCES

1. KLEIN F., *Vergleichende Betrachtungen über neuere geometrische Forschungen (Erlangen-Programm)*. Verlag A. Deichert, Erlangen, 1872.
2. WIGNER E., Symmetry and conservation laws. *Physics Today* **17**, 34–40, 1964.
3. MARKHASHOV L. M., On conformally invariant motions of a material point. *Prikl. Mat. Mekh.* **30**, 3–13, 1966.
4. MARKHASHOV L. M., On relativistic analogues of the dynamics of a material point. *Prikl. Mat. Mekh.* **53**, 382–395, 1989.
5. MARKHASHOV L. M., On a possible alternative in relativistic mechanics. In *Proceedings of the Fifth All-Union Conference on Analytical Mechanics, Theory of Stability and Control of Motion*. Vychisl. Tsentr, Akad. Nauk SSSR, Moscow, 1990.

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REVERSIBLE SYSTEMS. STABILITY AT 1:1 RESONANCE†

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Some general properties of reversible systems are studied: the nature of the stability of the trivial equilibrium position, the conditions for the existence of certain periodic solutions and symmetry of the phase portrait. It is shown that a discrete automorphism (symmetry) group generates integral manifolds. A detailed investigation of the stability of the trivial solution at 1:1 resonance is presented. The necessary and sufficient conditions for the stability of a model system are obtained and it is shown that instability of the system implies instability of the complete system.

1. SOME PROPERTIES OF REVERSIBLE SYSTEMS

CONSIDER an autonomous system of differential equations

$$dx_s/dt = f_s(x_1, \ldots, x_n) \quad (s = 1, 2, \ldots, n) \tag{1.1}$$

with smooth right-hand sides, whose phase flow is reversible [1, p 115]: there exists a non-degenerate linear mapping

$$\mathbf{M}\colon\ \mathbf{X} \to \mathbf{X},\quad t \to -t \tag{1.2}$$

such that

$$\mathbf{f}(\mathbf{x}) = -\mathbf{M}^{-1}\mathbf{f}(\mathbf{M}\mathbf{x}) \tag{1.3}$$

$$\mathbf{f} = (f_1, \ldots, f_n),\quad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}$$

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where $\mathbf{X}$ is the phase space. The mapping (1.2), (1.3) is a linear automorphism of system (1.1) [2]. Obviously, $\mathbf{M}^p:\mathbf{X}\to\mathbf{X}$, $t\to(-1)^p t$, where p is any integer, is also an automorphism of Eqs (1.1). The set ω of all such mappings is the cyclic group of symmetries generated by the operator (1.2), (1.3). The group ω is of order κ if and only if $\mathbf{M}^\kappa=\mathbf{E}$; otherwise it will be infinite.

The exponent κ may take only even values.

Indeed, let $\mathbf{A}$ be the matrix of the linear part. Then $\mathbf{MA}=-\mathbf{AM}$. It is known [3] that this equation has a non-trivial solution $\mathbf{A}$ if and only if $\mathbf{M}$ and $-\mathbf{M}$ have eigenvalues in common. For a κ-periodic matrix $\mathbf{M}$ ($\mathbf{M}^\kappa=\mathbf{E}$) this is possible only provided that $\kappa=2k$, $k\in\mathbf{Z}$.

Let $\mathbf{x}=\mathbf{x}(t,\mathbf{x}_0)$ be an integral curve of Eqs (1.1)–(1.3). Then the family

$$\mathbf{y}_p(t,\mathbf{x}_0)=\mathbf{M}^{-p}\mathbf{x}((-1)^p t,\mathbf{x}_0);\qquad p=0,1,\ldots,\kappa-1 \tag{1.4}$$

consists of integral curves of the system. Since Eqs (1.1) have a unique solution, the equality $\mathbf{M}^{-p}\mathbf{x}(0,\mathbf{x}_0)=\mathbf{x}(0,\mathbf{M}^{-p}\mathbf{x}_0)$ implies $\mathbf{y}_p(t,\mathbf{x}_0)=\mathbf{x}(t,\mathbf{M}^{-p}\mathbf{x}_0)$. We have thus proved the following lemma.

Lemma. The linear automorphism (1.2), (1.3) generates a free cyclic group of automorphisms, or a cyclic group of finite but even order κ (in the case $\mathbf{M}^\kappa=\mathbf{E}$). If $\mathbf{x}=\mathbf{x}(t\,\mathbf{x}_0)$ is an integral curve of system (1.1)–(1.3), then any curve of the family (1.4) is also an integral curve, and moreover $\mathbf{y}_p(t,\mathbf{x}_0)=\mathbf{x}(t,\mathbf{M}^{-p}\mathbf{x}_0)$.

We will now show that if the phase flow is reversible, the trivial equilibrium position cannot be asymptotically stable.

Theorem 1. The solution $\mathbf{x}=0$ of a reversible system (1.1)–(1.3) cannot be asymptotically stable.

Proof. Let us suppose that the trivial solution is asymptotically stable. Then there exists an integral curve $\mathbf{x}(t,\mathbf{x}_0)$ that has the property $\mathbf{x}(t,\mathbf{x}_0)\to 0$ as $t\to+\infty$. Obviously, $\mathbf{y}_1(t,\mathbf{x}_0)\to 0$ as $t\to-\infty$, contrary to the assumption of asymptotic stability.

Note that if $\mathbf{x}_0\in\mathbf{L}_p^{(1)}$, where $\mathbf{L}_p^{(1)}$ is the fixed set of the operator $\mathbf{M}^{-p}$, then $\mathbf{y}_p(t,\mathbf{x}_0)=\mathbf{x}(t,\mathbf{x}_0)$. Clearly, for any $p\in\mathbf{Z}$,

$$\mathbf{L}_p^{(1)}=\mathbf{L}(\bigcup_k \mathbf{L}_1^{(\lambda_k)}),\qquad \lambda_k^p=1,\quad k\in\mathbf{Z} \tag{1.5}$$

where $\mathbf{L}_1^{(\lambda_k)}$ is the eigensubspace of the operator $\mathbf{M}^{-1}$ belonging to the eigenvalue λ_k and $\mathbf{L}$ denotes linear span. If all the eigenvalues of the matrix $\mathbf{M}^{-1}$ have the values $\kappa\sqrt{1}$ (the case $\mathbf{M}^{-\kappa}=\mathbf{E}$, κ even), then $\lambda_k=\cos(2k\pi/\kappa)+i\sin(2k\pi/\kappa)$, where $1\leqslant p\leqslant\kappa-1$, $0\leqslant k\leqslant\kappa-1$, $pk\equiv(\mathrm{mod}\,\kappa)$. In the case of a free group ($\kappa=\infty$), the representation (1.5) remains valid, since the Jordan form of the matrix $\mathbf{M}^{-p}$ duplicates that of $\mathbf{M}^{-1}$ apart from the substitution $\lambda_k\to\lambda_k^p$.

Let us assume that p is even. Then $\mathbf{M}^{-p}\mathbf{x}(t)=\mathbf{x}(t)$, and therefore $\mathbf{L}_p^{(1)}$ is an integral manifold of system (1.1). But if $p=2r+1$, then $\mathbf{M}^{-p}\mathbf{x}(-t)=\mathbf{x}(t)$. Hence it follows that if $\mathbf{L}_p^{(1)}$ is an integral manifold, all its points are equilibrium positions, since any pair of solutions $\mathbf{x}(t)$, $\mathbf{x}(t+\text{const})$ that belongs to $\mathbf{L}_p^{(1)}$ consists of even functions of time.

Suppose that $\mathbf{L}_p^{(1)}$ is not an integral manifold. Then any trajectory that intersects $\mathbf{L}_p^{(1)}$ at two distinct points corresponding to successive instants of time $t_1^{(p)}$, $t_2^{(p)}$ will be called $\mathbf{L}_p^{(1)}$-normal [4]. It is clear that an $\mathbf{L}_p^{(1)}$-normal trajectory is periodic. We have thus proved the following theorem.

Theorem 2. Let ω be a cyclic group of automorphisms of order $\kappa\geqslant 2$. Then, if p is even, each of the non-trivial sets (1.5) is an integral manifold. If p is odd and $\mathbf{L}_p^{(1)}$ is not an integral set, then any $\mathbf{L}_p^{(1)}$-normal trajectory is closed and is periodic with period $T=2|t_1^{(p)}-t_2^{(p)}|$; if $\mathbf{L}_p^{(1)}$ is an integral manifold, all its points are equilibrium positions.

If $\kappa=2$ the integral set $\mathbf{L}_2^{(1)}$ is trivial; $\mathbf{L}_2^{(1)}=\mathbf{X}$. That any $\mathbf{L}_1^{(1)}$-normal trajectory is periodic is a known fact [4]. Conditionally periodic solutions were studied in [5].

In mechanical problems $\kappa=2$, i.e. $\mathbf{M}$ is an involution: $\mathbf{M}^2=\mathbf{E}$ [4, 6, 7]. In that case the canonical form of $\mathbf{M}$ is

$$\mathbf{M} = \begin{Vmatrix} \mathbf{E}_l & \mathbf{0} \\ \mathbf{0} & -\mathbf{E}_m \end{Vmatrix} \quad (l+m=n) \tag{1.6}$$

($\mathbf{E}_j$ is the identity matrix of order j). It follows that $\mathbf{M}$ is an orthogonal mapping.

The following proposition follows from the lemma and the properties of an involutory orthogonal mapping $\mathbf{M}$ [8].

Corollary. If system (1.1) admits of a linear involutory automorphism (1.2), (1.3), (1.6), then its phase portrait is symmetrical either about the origin (if $l=0$) or about the l-plane $\mathbf{L}_1^{(1)}$ ($l>0$, $m>0$), which is an eigensubspace of $\mathbf{M}$.

Note that if $\det\mathbf{M}=-1$ (improper transformation), the symmetry about $\mathbf{L}_1^{(1)}$ is a mirror reflection, since it reverses the orientation of the space.

Example 1. Consider the equations of motion of a holonomic mechanical system with positional forces and steady constraints:

$$\begin{gathered} \frac{d}{dt}\left(\frac{\partial T}{\partial q_j^{\cdot}}\right) - \frac{\partial T}{\partial q_j} = Q_j(\mathbf{q}), \quad Q_j(0)=0 \quad (j=1,2,\ldots,n) \\ 2T = \sum_{i,j} a_{ij}(\mathbf{q}) q_i^{\cdot} q_j^{\cdot}, \quad q=(q_1,\ldots,q_n) \end{gathered} \tag{1.7}$$

where $\mathbf{q}$ is the vector of generalized coordinates, T is the kinetic energy and Q_j are the generalized forces. It is known [9, 10] that the phase flow of system (1.7) is reversible, since Eqs (1.7) are invariant under the substitution $\mathbf{q}\to\mathbf{q}$, $\mathbf{q}^{\cdot}\to-\mathbf{q}^{\cdot}$, $t\to-t$. Hence it follows that, first, the phase portrait is symmetrical about the coordinate plane $\mathbf{q}^{\cdot}=0$ and second, the equilibrium position $\mathbf{q}=\mathbf{q}^{\cdot}=0$ cannot be asymptotically stable, whatever the positional forces acting on the system.

If the frequency equation

$$\det\| c_{ij} + \omega^2 a_{ij}(0) \| = 0, \quad c_{ij} = (\partial Q_i/\partial q_j)_0$$

has negative or complex solutions ω^2 then, by the Hadamard–Perron Theorem [11], Eqs (1.7) possess smooth integral manifolds $\mathbf{W}^s$ (stable), $\mathbf{W}^u$ (unstable) and $\mathbf{W}^c$ (central) that pass through the origin; moreover, $\mathbf{W}^u$ is the image of $\mathbf{W}^s$ under reflection in the plane $\mathbf{q}^{\cdot}=0$ and vice versa. The set $\mathbf{W}_0=\mathbf{W}^s\cap\mathbf{W}^u$ of all solutions doubly asymptotic to zero is mapped onto itself. Any trajectory that contains two points with zero velocity is periodic and symmetrical about the plane $\mathbf{q}^{\cdot}=0$.

2. 1:1 RESONANCE. LINEAR NORMALIZATION

A reversible system (1.1)–(1.3) satisfying the additional condition (1.6) may be written in the form

$$\begin{gathered} \mathbf{u}_*^{\cdot} = \mathbf{U}(\mathbf{u}_*,\mathbf{v}_*), \quad \mathbf{v}_*^{\cdot} = \mathbf{V}(\mathbf{u}_*,\mathbf{v}_*); \quad \mathbf{u}_*\in\mathbf{R}^l, \quad \mathbf{v}_*\in\mathbf{R}^m, \quad l+m=n \\ \mathbf{U}(\mathbf{u}_*,-\mathbf{v}_*) = -\mathbf{U}(\mathbf{u}_*,\mathbf{v}_*), \quad \mathbf{V}(\mathbf{u}_*,-\mathbf{v}_*) = \mathbf{V}(\mathbf{u}_*,\mathbf{v}_*) \end{gathered} \tag{2.1}$$

Let us assume that $\mathbf{U}$ and $\mathbf{V}$ are holomorphic functions of $\mathbf{u}_*$, $\mathbf{v}_*$ and that $l\geqslant m$. The equations of the first approximation are

$$\mathbf{u}_*^{\cdot} = \mathbf{A}_*\mathbf{v}_*, \quad \mathbf{v}_*^{\cdot} = \mathbf{B}_*\mathbf{u}_* \tag{2.2}$$

($\mathbf{A}_*$, $\mathbf{B}_*$ are constant matrices). If $l>m$ the matrix $\mathbf{D}$ of system (2.2) is singular, since it contains a square submatrix of zeros, the sum of whose height and breadth exceeds n [12, p. 170]. The multiplicity κ of the zero eigenvalue of $\mathbf{D}$ is at least $l-m$:

$$\kappa = \dim \mathbf{N}_{\mathbf{D}^\kappa} \geqslant \dim \mathbf{N}_{\mathbf{D}} = n - \operatorname{rank}\mathbf{D} = n - \operatorname{rank}\mathbf{A}_* - \operatorname{rank}\mathbf{B}_* \geqslant l-m$$

($\mathbf{N}_{\mathbf{D}}$ is the kernel of $\mathbf{D}$).

Consider the case $\kappa=l-m$ (then $\operatorname{rank}\mathbf{A}_*=\operatorname{rank}\mathbf{B}_*=m$). The remaining eigenvalues of $\mathbf{D}$ may be grouped in pairs λ, $-\lambda$, since $\mathbf{D}$ and $-\mathbf{D}$ are similar, so that in a suitable basis $\mathbf{D} = \mathrm{diag}(\mathbf{0}, \mathbf{A}, -\mathbf{A})$ where $\mathbf{A}$ is a non-singular $m \times m$ matrix. Hence it follows that system (2.2) may be reduced to the form

$$\xi^{\cdot} = \mathbf{0}, \quad \mathbf{u}^{\cdot} = \mathbf{A}\mathbf{v}, \quad \mathbf{v}^{\cdot} = \mathbf{A}\mathbf{u} \quad (\xi \in \mathbf{R}^{l-m}; \ \mathbf{u}, \mathbf{v} \in \mathbf{R}^{m}) \tag{2.3}$$

since

$$\mathrm{diag}(\mathbf{A}, -\mathbf{A}) \sim \left\| \begin{matrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A} & \mathbf{0} \end{matrix} \right\|$$

The non-zero roots of the characteristic equation of system (2.3) are found from the equation

$$\det(\mathbf{C} - \lambda^2 \mathbf{E}) = 0, \quad \mathbf{C} = \mathbf{A} \cdot \mathbf{A} = \| c_{ij} \| \tag{2.4}$$

and split into pairs $\pm\lambda_s$ ($s = 1, 2, \ldots, m$). System (2.1) may be stable only if $\lambda_s^2 \leqslant 0$ ($s = 1, 2, \ldots, m$).

Let us assume that all the numbers λ_s^2 are negative and that at least one pair consists of equal numbers: $\lambda_1^2 = \lambda_2^2$ (1:1 resonance). Assuming that there are no other resonances of order at most four, we reduce the linear approximation system (2.3) to canonical form.

Since in this case $\mathrm{rank}\,\mathbf{A} = \mathrm{rank}\,\mathbf{C} = m$, our task is to transform the linear system

$$\mathbf{u}^{\cdot\cdot} = \mathbf{C}\mathbf{u} \tag{2.5}$$

Let $r = \mathrm{rank}\,\mathbf{C}^* = m - 1$, where $\mathbf{C}^* = \mathbf{C} - \lambda_1^2 \mathbf{E}$. Then the desired linear transformation $\mathbf{u}, \mathbf{u}^{\cdot} \to \mathbf{z}, \bar{\mathbf{z}}$ has the following form (writing only the first group of the whole set of complex-conjugate transformations):

$$\begin{aligned} z_1 &= \sum_{j=1}^{m} p_{1j}(u_j^{\cdot} + \lambda_1 u_j) + i\mu \sum_{j=1}^{m} p_{2j} u_j \\ z_s &= \sum_{j=1}^{m} p_{sj}(u_j^{\cdot} + \lambda_s u_j) \quad (s = 2, 3, \ldots, m) \end{aligned} \tag{2.6}$$

Here $\mu = 1$ and the matrix $\mathbf{P} = \|p_{sj}\|$ has purely imaginary elements that satisfy systems of linear equations

$$(\mathbf{C} - \lambda_s^2 \mathbf{E})^{\mathrm{T}} \mathbf{p}_s^{\mathrm{T}} = 2i\mu\lambda_1 \delta_{1s} \mathbf{p}_2^{\mathrm{T}} \quad (s = 1, 2, \ldots, m) \tag{2.7}$$

where $\mathbf{p}_s = (p_{s1}, \ldots, p_{sm})$, and δ_{js} is the Kronecker delta. By (2.4), the determinants of these systems vanish, so that if $s \neq 1$ Eqs (2.7) have non-trivial solutions. If $s = 1$ the equations also have a non-trivial solution $\mathbf{p}_1^{\mathrm{T}}$ in the set $\mathbf{N}_2 \backslash \mathbf{N}_1$, where $\mathbf{N}_1$, $\mathbf{N}_2$ are the kernels of the operators $\mathbf{C}^{*\mathrm{T}}$, $(\mathbf{C}^{*\mathrm{T}})^2$, respectively. Indeed, $\mathrm{rank}(\mathbf{C}^{*\mathrm{T}})^2 = m - 2$, therefore $\mathbf{N}_2 \backslash \mathbf{N}_1$ is not empty, and $\mathbf{C}^{*\mathrm{T}}: \mathbf{N}_2 \backslash \mathbf{N}_1 \to \mathbf{N}_1$.

The matrix $\mathbf{P}$ is non-singular.

Clearly, the last $m-1$ rows of this matrix are independent, since they are eigenvectors of $\mathbf{C}^{\mathrm{T}}$ that belong to pairwise distinct eigenvalues $\lambda_2^2, \ldots, \lambda_m^2$. Suppose that $\mathbf{p}_1 = \alpha_2 \mathbf{p}_2 + \ldots \alpha_m \mathbf{p}_m$, where α_k are real numbers. The first equation of system (2.7) is

$$(\alpha_2 \mathbf{C}^{*\mathrm{T}} - 2i\lambda_1 \mathbf{E}) \mathbf{p}_2^{\mathrm{T}} = -\mathbf{C}^{*\mathrm{T}} \sum_{k=3}^{m} \alpha_k \mathbf{p}_k^{\mathrm{T}}$$

The determinant of the left-hand side is not zero, so that the vectors $\mathbf{p}_2, \ldots, \mathbf{p}_m$ are dependent, which is impossible.

Applying the transformations (2.6) to system (2.3), we obtain the following complex representation:

$$\begin{aligned} &\xi^{\cdot} = \mathbf{0}, \quad z_1^{\cdot} = \lambda_1 z_1 + i z_2, \quad \bar{z}_1^{\cdot} = -\lambda_1 \bar{z}_1 - i\bar{z}_2 \\ &z_s^{\cdot} = \lambda_s z_s, \quad \bar{z}_s^{\cdot} = -\lambda_s \bar{z}_s \quad (s = 2, 3, \ldots, m) \end{aligned}$$

The inverse transformation may be found from the equations (summation is from $j = 1$ to $j = m$)

$$\Sigma(p_{1j}\lambda_1+i\mu p_{2j})u_j=\tfrac{1}{2}(z_1+\bar z_1),\quad \Sigma p_{1j}u_j^{\cdot}=\tfrac{1}{2}(z_1-\bar z_1)$$
$$\Sigma p_{sj}u_j=\tfrac{1}{2}\lambda_s^{-1}(z_s+\bar z_s),\quad \Sigma p_{sj}u_j^{\cdot}=\tfrac{1}{2}(z_s-\bar z_s)\quad (s=2,3,\ldots,m)$$

Hence it follows that $\mathbf{u}$ and $\mathbf{v}=\mathbf{A}^{-1}\mathbf{u}^{\cdot}$ are linear combinations of the real and imaginary expressions $\mathbf{z}+\bar{\mathbf{z}}$ and $\mathbf{z}-\bar{\mathbf{z}}$, respectively.

If $r=m-2$, system (2.3) becomes

$$\xi^{\cdot}=\mathbf{0},\quad z_s^{\cdot}=\lambda_s z_s,\quad \bar z_s^{\cdot}=-\lambda_s\bar z_s\quad (s=1,2,\ldots,m) \tag{2.8}$$

by application of the transformations (2.6), (2.7), in which we must put $\mu=0$, $p_{1,m-1}=p_{2m}=1$, $p_{1m}=p_{2,m-1}=0$.

3. INSTABILITY. THE CASE OF NON-SIMPLE ELEMENTARY DIVISORS

Let us investigate the stability of the trivial solution of system (2.1) at 1:1 resonance. This problem was solved for Hamiltonian systems with two degrees of freedom in [13, 14]. A major role in the solution of the problem is played by the resonance subsystem [15], so we therefore first consider the case $l=m=2$. We introduce the notation

$$\rho_1=z_1\bar z_1,\quad \rho_2=z_2\bar z_2,\quad x=i(\bar z_1z_2-z_1\bar z_2),\quad x_1=i(\bar z_1\bar z_2-z_1z_2)$$

Normalizing up to terms of the third order inclusive, we obtain the following system:

$$\begin{aligned}
&\rho_1^{\cdot}=x[\mu+(B_{11}-C_{12})\rho_1+B_{12}\rho_2+C_{11}y]+O((\rho_1+\rho_2)^{5/2})\\
&\rho_2^{\cdot}=-x[A_{21}\rho_1+(A_{22}-C_{21})\rho_2+C_{22}y]+O((\rho_1+\rho_2)^{5/2})\\
&x^{\cdot}=R(\rho_1,\rho_2,y)+O((\rho_1+\rho_2)^{5/2})\\
&x_1^{\cdot}=R_1(z_1,z_2,\bar z_1,\bar z_2)+O((\rho_1+\rho_2)^{5/2})\\
&R=2[\mu\rho_2-A_{21}\rho_1^2+(B_{11}-A_{22}-C_{12}+C_{21})\rho_1\rho_2+B_{12}\rho_2^2]+\\
&+y[(A_{11}-B_{21}-C_{22})\rho_1+(A_{12}-B_{22}+C_{11})\rho_2]+(C_{12}-C_{21})y^2\\
&y=\bar z_1z_2+z_1\bar z_2,\quad x^2+y^2=4\rho_1\rho_2
\end{aligned} \tag{3.1}$$

where R_1 is a fourth-degree polynomial whose exact form is immaterial, A_{ij}, B_{ij}, C_{ij} are real coefficients. We have $\mu=1$ if the elementary divisors are not simple and $\mu=0$ if they are.

Theorem 3. The trivial solution of system (2.1) at 1:1 resonance is unstable in Lyapunov's sense if the function $R(\rho_1,\rho_2,y)$ has a fixed sign in a sufficiently small neighbourhood of the origin lying entirely within the cone $\rho_1\geqslant 0$, $\rho_2\geqslant 0$.

The truth of this statement for $m=l=2$ follows from the fact that x is a Chetayev function for system (3.1) if R is sign-definite for $\rho_1\geqslant 0$, $\rho_2\geqslant 0$ (the order of R is bounded above by the degree of homogeneity $\rho_1^2+\rho_2^2$). For arbitrary l and m the Chetayev function is constructed in the form

$$V=x^2-\gamma^2\sum_{j=1}^{l-m}\xi_j^2-\sum_{s=3}^{m}z_s\bar z_s$$

where γ is a suitably chosen constant; instability is also derived from the fact that R has a fixed sign. Therefore, if $\mu=1$, it follows from the inequalities

$$|\rho_1y|\leqslant 2\rho_1^{3/2}\rho_2^{1/2},\quad \rho_2+|A_{21}|\rho_1^2\geqslant 2(|A_{21}|\rho_2)^{1/2}\rho_1$$

that $A_{21}<0$ is a sufficient condition for instability.

Corollary.† If $l=m$ and the matrix $\mathbf{C}$ has non-simple elementary divisors, then the equilibrium position is unstable in Lyapunov's sense, provided that $A_{21}<0$.

†KHAZIN L. G., On resonance instability of the equilibrium position in the case of multiple frequencies. Preprint No. 97, Inst. Prikl. Mat., Akad. Nauk SSSR, 1975.

4. STABILITY IN THE CASE OF NON-SIMPLE ELEMENTARY DIVISORS

We shall now show that if $\mu = 1$ and $A_{21} < 0$, the trivial solution of the model system obtained from (3.1) by dropping all terms $O[(\rho_1 + \rho_2)^{5/2}]$ and the equation for x_1 is stable.

Let $l = m = 2$. The model is reversible, with a linear automorphism $x \to -x$, $y \to y$, $\rho_1 \to \rho_1$, $\rho_2 \to \rho_2$. We will first consider the behaviour of a trajectory along which x vanishes at most once. Suppose that at time t_0 the values of ρ_1, ρ_2 satisfy the condition $\rho_{10} + \rho_{20} \leqslant \delta^2$ (δ is some small positive number) but at $t > t_0$ x preserves its sign as long as ρ_1, ρ_2 remain in a σ-neighbourhood $\rho_1 + \rho_2 < \sigma$ ($\sigma > \delta$). Note that since the phase portrait is symmetrical, the case $t < t_0$ reduces to that considered here.

If $x \neq 0$, we conclude from the first two equations of the model system that

$$d\rho_2/d\rho_1 = f(\rho_1, \rho_2, y), \quad |f(\rho_1, \rho_2, y)| \leqslant k(\rho_1 + \rho_2) \quad (k = \text{const})$$

Therefore, in the domain $\rho_1 + \rho_2 \leqslant \delta$, the increments of the variables ρ_1 and ρ_2 satisfy the inequality $|\Delta\rho_2| \leqslant k\delta|\Delta\rho_1|$. On the boundary $\rho_1 + \rho_2 = \delta$ we have

$$|\Delta\rho_2| \leqslant k\delta^2, \quad \rho_2 \leqslant \rho_{20} + |\Delta\rho_2| \leqslant (1+k)\delta^2, \quad \rho_1 = \delta - \rho_2 \geqslant \delta - (1+k)\delta^2$$

Hence it follows that if $x < 0$ the trajectory does not hit the boundary of the δ-neighbourhood, since $\rho_1^{\bullet} < 0$. If $x > 0$, then the quantities ρ_1^2 and ρ_2 are of the same order on the boundary of the neighbourhood (if it is reached), irrespective of the choice of initial data. Since $\rho_1^{\bullet} > 0$, $\rho_2^{\bullet} < 0$, as the motion continues we obtain

$$\rho_1^2 \geqslant [1 - (1+k)\delta]^2 (1+k)^{-1}\rho_2.$$

At the same time,

$$d\rho_2/d\rho_1 = -A_{21}\rho_1 + f^*(\rho_1, \rho_2, y), \quad |f^*| \leqslant k^*\rho_1^{3/2} \quad (k^* = \text{const} > 0)$$

Let us assume that ρ_1 increases together with $\epsilon_1 \geqslant \delta - (1+k)\delta^2$, up to $\epsilon = \alpha\epsilon$ ($\alpha = \text{const} > 1$). Then $\Delta\rho_1 = (\alpha - 1)\epsilon_1$, $\Delta\rho_1^2 = (\alpha^2 - 1)\epsilon_1^2$ and $\Delta\rho_2 \leqslant -\frac{1}{2}A_{21}(\alpha^2 - 1)\epsilon_1^2 + k^*(\alpha^{5/2} - 1)\epsilon_1^{5/2}$. Consequently,

$$\rho_2 \leqslant (1+k)\delta^2 + \Delta\rho_2 < \tfrac{1}{2}[(1+k) - A_{21}(\alpha^2 - 1)]\epsilon_1^2 + k^*(\alpha^{5/2} - 1)\epsilon_1^{5/2}$$

If $\alpha^2 > 2(1+k)A_{21}^{-1} + 1$, then for sufficiently small ϵ_1 we have $\rho_2 < 0$, which is impossible. Thus none of the trajectories under consideration can reach the boundary of the ϵ-neighbourhood, if the initial conditions belong to a δ^2-neighbourhood (δ is the minimum root of the equation $\epsilon/\alpha = \delta - (1+k)\delta^2$).

Let us consider trajectories on which x vanishes at least twice. By Theorem 2, these trajectories are closed curves. The family of these periodic solutions $\{\eta(t) = (x(t), \rho_1(t), \rho_2(t))\}$ does not lead to instability.

Suppose the contrary: for some $\epsilon > 0$ the family intersects the sphere S_ϵ in a sequence of points $\{\eta'_{0k}\}$. Let $\{\eta_{0k}\}$ be a convergent subsequence (the sphere S_ϵ is compact), and $\{t_k\}$ the corresponding sequence of times, which satisfies the conditions

$$\{t_k\} \to -\infty, \quad \eta(\eta_{0k}, t_k) = \min_{-T_k \leqslant t < 0} \|\eta(\eta_{0k}, t)\| \to 0 \quad \text{as} \quad k \to \infty$$

[T_k is the period of $\eta(\eta_{0k}, t)$]. Let $\eta_0 = \lim_{k\to\infty} \eta_{0k}$. Obviously $\lim \eta(\eta_0, t_k) = 0$ as $k \to \infty$. This means that the solution $\eta(\eta_0, t)$ leads to instability. The function x vanishes along $\eta(\eta_0, t)$ at most once, for otherwise $\eta(\eta_0, t)$ would be a periodic function of time and so the phase point would reach $\eta = 0$ in a finite time, which is impossible. This result implies a contradiction: on the one hand, as follows from previous arguments, the trajectory $\eta(\eta_0, t)$ cannot leave the ϵ-neighbourhood; on the other hand, it must. This completes the proof of stability.

Combining these results, we obtain the following theorem.

Theorem 4. Let $A_{21} \neq 0$, $l = m = 2$, and assume that **C** has non-simple elementary divisors. A necessary and sufficient condition for the model system to be stable is that $A_{21} > 0$.

5. THE CASE OF SIMPLE ELEMENTARY DIVISORS

Let $l = m = 2$. The third-approximation model system is

$$\mathbf{z}^{\cdot} = x\mathbf{A}\mathbf{z}, \quad \mathbf{z} = (\rho_1, \rho_2, y)^{\mathrm{T}}, \quad x = \pm\sqrt{4\rho_1\rho_2 - y^2} \tag{5.1}$$

$$\mathbf{A} = \| a_{ij} \|_1^3, \quad a_{11} = B_{11} - C_{12}, \; a_{12} = B_{12}, \; a_{13} = C_{11}, \; a_{21} = -A_{21}$$
$$a_{22} = C_{21} - A_{22}, \; a_{23} = -C_{22}, \; a_{31} = B_{21} - A_{11} - C_{22}, \; a_{32} = B_{22} + C_{11} - A_{12}$$
$$a_{33} = C_{21} - C_{12}.$$

The cone

$$\mathbf{K} = \{\rho_1, \rho_2, y: \; 4\rho_1\rho_2 \geqslant y^2, \; \rho_1 \geqslant 0, \rho_2 \geqslant 0\} \tag{5.2}$$

is an integral set, since by (3.1) the inequality $4\rho_1\rho_2 \geqslant y^2$ holds throughout the motion. On the hyperboloid conic surface $y^2 = 4\rho_1\rho_2$ the function x vanishes, changing sign, and therefore the boundary of **K** reflects the phase curves of system (5.1), inducing the phase point to perform retrograde motion. The surface $y^2 = 4\rho_1\rho_2$ is singular, because the solution fails to be unique there. Note that the reflection of the phase curves (5.1) is due to reversibility of the phase flow.

It is clear that the phase point of system (5.1) moves along the phase curves of the linear system

$$\mathbf{z}_*^{\cdot} = \mathbf{A}\mathbf{z}_* \tag{5.3}$$

If the point is reflected twice, this corresponds to periodic motion along some part of the phase curve (5.3) that belongs to **K**.

We will now derive the necessary and sufficient conditions for the existence of singular directions (invariant rays):

$$\rho_2 = k\rho_1, \quad y = k_1\rho_1, \quad k_1^2 < 4k \tag{5.4}$$

where k_1, $k_2 > 0$ are constant parameters. To that end, we define

$$G_3 = C_{11}^2(B_{22} - A_{12} + C_{11}) + B_{12}(B_{12}C_{22} + C_{11}C_{12} - C_{11}A_{22})$$
$$G_2 = C_{11}^2(B_{21} - A_{11} - C_{22}) + 2C_{11}C_{22}(B_{22} - A_{12} + C_{11}) + B_{12}(B_{11}C_{22} - C_{22}C_{21} -$$
$$- C_{11}A_{21}) + (B_{11} - C_{12} + A_{22} - C_{21})(B_{12}C_{22} + C_{12}C_{11} - C_{11}A_{22})$$
$$G_1 = 2C_{11}C_{22}(B_{21} - A_{11} - C_{22}) + C_{22}^2(B_{22} - A_{12} + C_{11}) + A_{21}(B_{12}C_{22} + C_{11}C_{12} -$$
$$- C_{11}A_{22}) + (B_{11} - C_{12} + A_{22} - C_{21})(B_{11}C_{22} - C_{22}C_{21} - C_{11}A_{21})$$
$$G_0 = C_{22}^2(B_{21} - A_{11} - C_{22}) + A_{21}(B_{11}C_{22} - C_{22}C_{21} - C_{11}A_{21})$$
$$A_1 = -[k(B_{11} + A_{22} - C_{12} - C_{21}) + k^2B_{12} + A_{21}]\,[2\sqrt{k}(kC_{11} + C_{22})]^{-1}$$

Suitable computations prove the following theorem.

Theorem 5. Equations (5.1) admit of particular solutions of the form (5.4) if and only if there exists a positive number k such that both of the following conditions hold:

$$G_3k^3 + G_2k^2 + G_1k + G_0 = 0, \quad |A_1| < 1 \tag{5.5}$$

Under the conditions, $k_1 = 2A_1\sqrt{k}$.

In the limiting case, $k_1^2 = 4k(|A_1| = 1)$, the invariant ray lies on the boundary of **K**, and it is observed to degenerate, disintegrating into an infinite set of equilibrium positions.

We assert that in the non-degenerate case the system is stable provided that it has no invariant rays (5.4).

Let us consider the structurally stable situation, in which $\mathbf{z}_* = 0$ is a hyperbolic singular point of Eqs (5.3). The characteristic equation will then have no roots with identical real parts (excluding the case of complex-conjugate eigenvalues). There are ten possible relative positions of the eigenvalues λ_1, λ_2, λ_3:

$$
\begin{aligned}
&1)\ \lambda_1 < \lambda_2 < \lambda_3 < 0;\quad 2)\ \lambda_1 < \lambda_2 < 0,\ \ \lambda_3 > 0\\
&3)\ \lambda_1 = \bar{\lambda}_2,\ \mathrm{Re}\,\lambda_1 < \lambda_3 < 0;\quad 4)\ \lambda_1 = \bar{\lambda}_2,\ \lambda_3 < \mathrm{Re}\,\lambda_1 < 0\\
&5)\ \lambda_1 = \bar{\lambda}_2,\ \ \mathrm{Re}\,\lambda_1 < 0,\ \ \lambda_3 > 0
\end{aligned}
\tag{5.6}
$$

The other five case are obtained from (5.6) by the substitution $\lambda_j \to -\lambda_j$. The phase portrait of Eqs (5.3) is readily visualized in three dimensions (see for example [6]).

The fact that system (5.1) has no invariant rays means that neither the real eigenvectors $\boldsymbol{\xi}_k$ of $\mathbf{A}$ nor the reverse vectors $-\boldsymbol{\xi}_k$ lie on the cone $\mathbf{K}$. Examining the form of the general solution of Eqs (5.3),

$$
\mathbf{z}_*(t) = \sum_{k=1}^{3} C_k e^{\lambda_k t} \boldsymbol{\xi}_k \tag{5.7}
$$

we see that $\mathbf{K}$ contains no positive (negative) semi-trajectories of Eqs (5.3), since the phase point leaves $\mathbf{K}$ as $t \to +\infty$ and as $t \to -\infty$.

Indeed, in cases (1) and (2) this follows from the asymptotic representation of $\mathbf{z}_*(t)$:

$$
\mathbf{z}_*(t) \sim C_k e^{\lambda_k t} \boldsymbol{\xi}_k \quad \text{as} \quad t \to +\infty \quad (t \to -\infty)
$$

(where ξ_k is a suitable eigenvector). In cases (3)–(5) the phase point $\mathbf{z}_*$ spirals around the straight line that contains ξ_3 and may therefore remain in $\mathbf{K}$ for a limited time only.

Thus if $\mathbf{K}$ does not contain invariant rays, the solutions that lie in $\mathbf{K}$ are periodic functions of time, and therefore the model system (5.1) is stable. The appearance of non-degenerate invariant rays leads to instability, which remains true in the full system (Theorem 3). Note that instability of the full system also follows from earlier results [17].†

The results of this discussion may be stated as a separate theorem. Let us say that the trivial solution of Eqs (5.3) is *structurally stable* if one of conditions (5.6) or the conditions obtained from them by substituting $\lambda_j \to -\lambda_j$ is valid.

Theorem 6. Assume that the trivial solution of Eqs (5.1) is structurally stable for system (5.3) and that $\partial\mathbf{K}$ does not contain invariant rays. Then the model system (5.1) is stable if and only if conditions (5.5) do not hold for any $k \in (0, +\infty)$; if the model system is unstable, the same holds for the full system.

6. EXAMPLE

Let us consider a mechanical system in a horizontal plane, consisting of two identical rods of mass m and length l, connected to one another and to a stationary base by ideal hinges and spiral springs of stiffness c_2 and c_1, respectively. We shall assume that a constant tracking force $\mathbf{F}$ is applied at the free end of the second rod, directed along its axis; in the natural configuration of the system the rods are undeformed.

This system may serve as a discrete model of an elastic rod driven by a tracking force. Adequate rod models of this type have indeed been used previously to investigate the behaviour of elastic beams, cables and ropes [18–20].

The motion of the system is described by Eqs (1.7) in which the generalized coordinates are the angular deviations of the rods from equilibrium, i.e. φ_1, φ_2, and

$$
\begin{aligned}
&T = {}^1/_6\, ml^2 [4\dot{\varphi}_1^2 + 3\dot{\varphi}_1\dot{\varphi}_2 \cos(\varphi_2 - \varphi_1) + \dot{\varphi}_2^2]\\
&Q_1 = -c_1\varphi_1 + c_2(\varphi_2 - \varphi_1) - Fl\sin(\varphi_2 - \varphi_1),\quad Q_2 = c_2(\varphi_1 - \varphi_2)
\end{aligned}
\tag{6.1}
$$

The characteristic equation of the linear approximation system is

$$
7\lambda^4 + \frac{6}{ml^2}(2c_1 + 16c_2 - 5Fl)\lambda^2 + \frac{36}{m^2 l^4} c_1 c_2 = 0 \tag{6.2}
$$

†See also: MEDVEDEV S. V., Proof of an instability lemma. Unpublished paper, VINITI, No. 1088-82, 1982.

Hence it follows [21, p. 212] that the system is stable to a first approximation if

$$a = 2c_1 + 16c_2 - 5Fl > 0, \quad a^2 - 28c_1c_2 > 0 \tag{6.3}$$

A few rigorous conclusions have been drawn [10] as to stability in this region. In particular, it has been shown that the parameter values satisfying the resonance relation 1:3 define unstable regimes. On the boundary of the region (6.3), that is, when $a^2 = 28c_1c_2$, the natural frequencies satisfy the resonance relation $\omega_1 = \omega_2$ if $a>0$, and then

$$\lambda_{1,2} = \lambda_{3,4} = \pm i\omega, \quad \omega^2 = \frac{3}{7}\frac{2c_1 + 16c_2 - 5Fl}{ml^2}, \quad \omega > 0$$

$$\mathbf{C}^* = \begin{Vmatrix} -\frac{3}{7ml^2}(2c_1 - 6c_2 + Fl) & \frac{6}{7ml^2}(5c_2 - 2Fl) \\ \frac{6}{7ml^2}(3c_1 + 11c_2 - 3Fl) & \frac{3}{7ml^2}(2c_1 - 6c_2 + Fl) \end{Vmatrix}$$

Clearly, rank $\mathbf{C}^* = 0$ only if $c_1 = c_2 = 0$, $F = 0$, so the matrix $\mathbf{C}$ has non-simple elementary divisors if it is non-singular. The matrix $\mathbf{P}$ of the linear transformation (2.6) is

$$\mathbf{P} = \begin{Vmatrix} \frac{2(5c_2 - 2Fl)}{\omega} i & \left(\frac{2c_1 - 6c_2 + Fl}{\omega} + \frac{14}{3} ml^2\omega\right) i \\ 2(5c_2 - 2Fl) i & (2c_1 - 6c_2 + Fl) i \end{Vmatrix}$$

Developing the right-hand sides of the equations of motion in series, we see that there are no terms of the second-order of smallness; the third-order terms in the equations for φ_1, φ_2 are

$$\Phi_1 = \frac{108}{49ml^2} c_1\varphi_1(\varphi_2 - \varphi_1)^2 + \frac{1}{49ml^2}(207c_2 - 94Fl)(\varphi_2 - \varphi_1)^3 + \frac{1}{7}(9\dot\varphi_1^2 + 6\dot\varphi_2^2)(\varphi_2 - \varphi_1)$$

$$\Phi_2 = -\frac{162}{49ml^2} c_1\varphi_1(\varphi_2 - \varphi_1)^2 + \frac{1}{49ml^2}(673c_2 - 246Fl)(\varphi_2 - \varphi_1)^3 - \frac{1}{7}(24\dot\varphi_1^2 + 9\dot\varphi_2^2)(\varphi_2 - \varphi_1)$$

Using the substitution inverse to (2.6), we can express the right-hand sides of (6.4) in terms of the new variables $z_j, \bar z_j$:

$$\Phi_1 = i\alpha_1 z_2 z_1 \bar z_1 + \ldots, \quad \Phi_2 = i\alpha_2 z_2 z_1 \bar z_1 + \ldots$$

It then follows from the second equation of (2.6), which is

$$\dot z_2 = p_{21}(\ddot\varphi_1 - i\omega\dot\varphi_1) + p_{22}(\ddot\varphi_2 - i\omega\dot\varphi_2)$$

that $A_{21} = p_{21}\alpha_1 + p_{22}\alpha_2$. Computations show that the coefficient A_2 is defined by

$$A_{21} = -\frac{p_{21} + p_{22}}{56\omega\Delta^3}\Big\{\frac{3}{49}\frac{p_{21} + p_{22}}{ml^2\omega^2}[(108p_{21} - 162p_{22})p_{22}c_1 - ((207c_2 - 94Fl)p_{21} +$$
$$+ (673c_2 - 246Fl)p_{22})(p_{21} + p_{22})] + (24p_{22} - 9p_{21})p_{12}^2 + (9p_{22} - 6p_{21})p_{11}^2\Big\}. \tag{6.4}$$

By Theorem 4, the inequality $A_{21} > 0$ is a necessary and sufficient condition for stability to the third order. If $A_{21} < 0$ we have instability in the rigorous non-linear setting. Analysis shows that, depending on the values of c_1, c_2, m, l, F, the coefficient A_{21} may be either positive or negative.

A separate discussion must be devoted to the case $5c_2 - 2Fl = 0$, which corresponds to the degeneracy $(\det\mathbf{P})^2 + (4c_1 - 7c_2)^2 = 0$. We have $c_1 = 7c$, $c_2 = 4c$, $Fl = 10c$. To determine the matrix $\mathbf{P}$ we again use Eqs (2.7), to get

$$\mathbf{P} = \begin{Vmatrix} i & \frac{ml^2}{30c} i \\ 0 & i \end{Vmatrix}$$

Noting that $\omega^2 = 12c/(ml^2)$, we find from (6.4) that $A_{21} = -2359/(94\omega) < 0$, implying that this equilibrium position is unstable in Lyapunov's sense.

REFERENCES

1. BIRKHOFF G. D., *Dynamical Systems*. American Mathematical Society, New York, 1927.
2. BRYUNO A. D., *The Local Method of Non-linear Analysis of Differential Equations*. Nauka, Moscow, 1979.
3. GANTMAKHER F. R., *Theory of Matrices*. Nauka, Moscow, 1988.
4. HEINBOCKEL J. N. and STRUBLE R. A., Periodic solutions for differential systems with symmetries. *J. SIAM* **13,** 425–440, 1965.
5. SEVRYUK M. B., Reversible systems. *Lecture Notes in Mathematics* **1211.** Springer, Berlin, 1986.
6. HALE J. K., *Oscillations in Nonlinear Systems*. McGraw-Hill, New York, 1963.
7. TKHAI V. N., The reversibility of mechanical systems. *Prikl. Mat. Mekh.* **55,** 578–586, 1991.
8. ROZENFELD B. A., *Multidimensional Spaces*. Nauka, Moscow, 1967.
9. WINTNER A., *Theoretical Foundations of Celestial Mechanics*. Princeton University Press, Princeton, 1947.
10. TKHAI V. N., The stability of mechanical systems driven by positional forces. *Prikl. Mat. Mekh.* **44,** 40–48, 1980.
11. MARSDEN L. E. and McCRACKEN M., *The Hopf Bifurcation and Its Applications*. Springer, New York, 1976.
12. PETROVSKII I. G., *Lectures on the Theory of Ordinary Differential Equations*. Nauka, Moscow, 1970.
13. SOKOL'SKII A. G., The stability of an autonomous Hamiltonian system with two degrees of freedom in the case of equal frequencies. *Prikl. Mat. Mekh.* **38,** 791–799, 1974.
14. KOVALEV A. M. and CHUDNENKO A. N., The stability of the position of equilibrium of a two-dimensional Hamiltonian system in the case of equal frequencies. *Dokl. Akad. Nauk UkrSSR, Ser. A* 11, 1010–1013, 1977.
15. KUNITSYN A. A. and MARKEYEV A. P., Stability in resonance cases. In Itogi Nauki i Tekhniki. Obshchaya Mekhanika (*Progress in Science and Technology. General Mechanics*), Vol. 4, pp. 58–139. VINITI, Moscow, 1979.
16. ARNOL'D V. I., *Ordinary Differential Equations*. Nauka, Moscow, 1975.
17. FURTA S. D., Asymptotic solutions of systems of differential equations in the case of purely imaginary eigenvalues. *Differents. Uravn.* **26,** 1348–1351, 1990.
18. HERRMAN G., Stability of equilibrium of elastic systems subjected to non-conservative forces. *Appl. Mech. Rev.* **20,** 103–108, 1967.
19. HUSTON R. L. and KAMMAN J. W., Validation of finite segment cable models. *Comput. Struct.* **15,** 653–660, 1982.
20. ZAKHAROVA Ye. Ye., KRASIL'NIKOV P. S. and MAKAROV B. V., Estimating the accuracy of the formation of the medical dose field of a tether intraband gamma-therapeutic apparatus. *Med. Tekhnika* 3, 5–9, 1990.
21. MERKIN D. R., *Introduction to the Theory of Stability of Motion*. Nauka, Moscow, 1976.

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POISSON STABILITY OF REVERSIBLE SYSTEMS†

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An investigation is presented of the stability in Poisson's sense of reversible systems in which the phase volume is not invariant, a particular example of which is non-holonomic systems. Criteria are proposed for the stability of such systems in Poisson's sense, and the existence of integral invariants is discussed.

1. CONSIDER an autonomous system of differential equations

$$d\mathbf{x}/dt = \mathbf{X}(\mathbf{x}) \tag{1.1}$$

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